The Yield To Maturity

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Yield to maturity (YTM) is the discount rate at which the sum of all future cash flows from the bond (coupon and principal payments) is equal to the current price of the bond. YTM is the internal rate of return (IRR) earned by an investor who buys the bond today at the market price (cash out) and receives all scheduled coupon and principal payments (cash in). Note that YTM is the rate earned by the investor assuming that the bond does not default.

In this white paper we will value a treasury bond and calculate it's yield to maturity. To that end we will use the following hypothetical problem...

Our Hypothetical Problem

The tables below present discount factors at time zero of $100 US Treasury Strips at different maturities and our treasury bond assumptions...

Table 1: US Treasury Strip Prices

Year	Price	Factor
0	100.00	1.0000
1	95.41	0.9541
2	90.66	0.9066
3	85.02	0.8502
4	80.30	0.8030
5	75.64	0.7564
6	70.89	0.7089
7	65.25	0.6525
8	60.23	0.6023
9	55.33	0.5533
10	50.63	0.5063

Table 2: Treasury Bond Assumptions

Description	Value
Face value	$ 1,000.00
Annual coupon rate	5.00%
Term in years	10.00

Question 1: What is the bond's value today?

Question 2: What is the bond's yield to maturity?

Bond Price Model

We will define the variables B_t to be bond price at time t, the variable F to be the bond's face value, the variable ϕ to be the continuous-time coupon rate, the variable D_t to be the discount factor at time t, and the variable T to be bond term in years. The equation for bond price at time zero is...

$$B_0 = \int\limits_0^T \phi \, F \, D_u \, \delta u + F \, D_T \tag{1}$$

Note that we are assuming that coupon payments are paid continuously, which is similar to the Black Scholes OMP that assumes that dividends are paid continuously.

We will model the discount factor at time t via the following parabola equation... [1]

$$D_t = a \, t^2 + b \, t + c \tag{2}$$

Using Equation (2) above we can rewrite Equation (1) above as...

$$B_0 = \phi\, F \int\limits_0^T \Big(a\, u^2 + b\, u + c\Big)\delta u + F\Big(a\, T^2 + b\, T + c\Big) \tag{3}$$

Using Appendix Equation (17) below the solution to Equation (3) above is...

$$B_0 = F\bigg[\phi\bigg(\frac{1}{3}\, a\, T^3 + \frac{1}{2}\, b\, T^2 + c\, T\bigg) + \Big(a\, T^2 + b\, T + c\Big)\bigg] \tag{4}$$

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We will define the variable ω to be the continuous-time yield to maturity. Note that we can rewrite bond price Equation (1) above as...

$$B_0 = \int\limits_0^T \phi\, F\, \mathrm{Exp}\bigg\{-\omega\, u\bigg\}\delta u + F\, \mathrm{Exp}\bigg\{-\omega\, T\bigg\} = F\bigg[\phi \int\limits_0^T \mathrm{Exp}\bigg\{-\omega\, u\bigg\}\delta u + \mathrm{Exp}\bigg\{-\omega\, T\bigg\}\bigg] \tag{5}$$

Using Appendix Equation (19) below the solution to Equation (5) above is...

$$B_0 = \phi\, F\left[\frac{1}{\omega}\bigg(1 - \mathrm{Exp}\bigg\{-\omega\, t\bigg\}\bigg) + \mathrm{Exp}\bigg\{-\omega\, T\bigg\}\right] = F\left[\frac{\phi}{\omega} - \frac{\phi}{\omega}\, \mathrm{Exp}\bigg\{-\omega\, T\bigg\} + \mathrm{Exp}\bigg\{-\omega\, T\bigg\}\right] \tag{6}$$

Using Appendix Equations (20), (21) and (22) below the equation for the derivative of Equation (6) above with respect to ω is...

$$\frac{\delta B_0}{\delta\, \omega} = F\left[-\frac{\phi}{\omega^2} + \phi\, \mathrm{Exp}\bigg\{-\omega\, T\bigg\}\bigg(1 + \frac{1}{\omega^2}\bigg) - \omega\, \mathrm{Exp}\bigg\{-\omega\, T\bigg\}\right] \tag{7}$$

We will define the function $f(\omega)$ to be the ratio of bond price at time zero to face value. Using Equation (6) above the equation for this function is...

$$f(\omega) = \frac{\phi}{\omega} - \frac{\phi}{\omega}\, \mathrm{Exp}\bigg\{-\omega\, T\bigg\} + \mathrm{Exp}\bigg\{-\omega\, T\bigg\} \tag{8}$$

We will define the function $f'(\omega)$ to be the derivative of the equation for the ratio of bond price at time zero to face value with respect to ω. Using Equation (7) above the equation for this function is...

$$f'(\omega) = -\frac{\phi}{\omega^2} + \phi\, \mathrm{Exp}\bigg\{-\omega\, T\bigg\}\bigg(1 + \frac{1}{\omega^2}\bigg) - \omega\, \mathrm{Exp}\bigg\{-\omega\, T\bigg\} \tag{9}$$

The yield to maturity (ω) in the equations above is unknown and must be solved for. Since those equations are nonlinear we will use the Newton-Raphson Method to solve for ω. If we define the variable $\hat{\omega}$ to be the guess value of ω and the variable ϵ to be the error term then we can set up the following equation... [2]

$$\hat{\omega} + \frac{f(\omega) - f(\hat{\omega})}{f'(\hat{\omega})} = \omega + e \tag{10}$$

To solve for ω we will iterate Equation (10) above until...

$$\text{if... } \hat{\omega} \approx \omega \text{ ...then... } f(\omega) - f(\hat{\omega}) \approx 0 \text{ ...and... } e \approx 0 \tag{11}$$

The Solution To Our Hypothetical Problem

Question 1: What is the bond's value today?

Our first task is to calculate the parameters to the parabola that models the discount factor. To do that we will use the market data in Table 1 above for years zero, five and ten. The matrix and vector definitions are therefore... [1]

$$\mathbf{A}=\begin{bmatrix}0 & 0 & 1\\ 5 & 25 & 1\\ 10 & 100 & 1\end{bmatrix}\text{ ...and... }\vec{\mathbf{u}}=\begin{bmatrix}a\\ b\\ c\end{bmatrix}\text{ ...and... }\vec{\mathbf{v}}=\begin{bmatrix}1.0000\\ 0.7564\\ 0.5063\end{bmatrix}\text{ ...such that... }\mathbf{A}\,\vec{\mathbf{u}}=\vec{\mathbf{v}} \tag{12}$$

Using the matrix and vector definitions in Equation (12) above we can solve for our parabola parameters as follows...

$$\mathbf{A}^{-1}\,\vec{\mathbf{v}}=\vec{\mathbf{u}}=\begin{bmatrix}-0.00013\\ -0.04806\\ 1.00000\end{bmatrix} \tag{13}$$

Since we are working in continuous-time we need to convert the bond's annual coupon rate from discrete-time to continuous-time. Using the parameters in Table 2 above the continuous-time coupon rate is...

$$\phi=\ln(1+0.05)=0.04879 \tag{14}$$

Using Equations (4), (13) and (14) above and the parameters in Table 2 above the value of the bond today is...

$$B_0=1000\times\Bigg[0.04879\times\left(\frac{1}{3}\times-0.00013\times10^3+\frac{1}{2}\times-0.04806\times10^2+1\times10\right)+$$
$$\left(-0.00013\times10^2-0.04806\times10+1\right)\Bigg]=874.85 \tag{15}$$

Question 2: What is the bond's yield to maturity?

The iteration of Equation (10) above (i.e. The Newton-Raphson Method) is as follows...

Iteration	$f(\omega)$	$\hat{\omega}$	$f(\hat{\omega})$	$f'(\hat{\omega})$	$f(\omega)-f(\hat{\omega})$	new $\hat{\omega}$
1	0.87485	0.04879	1.00000	-7.913	-0.1252	0.06461
2	0.87485	0.06461	0.88350	-5.571	-0.0087	0.06616
3	0.87485	0.06616	0.87294	-5.404	0.0019	0.06581
4	0.87485	0.06581	0.87533	-5.441	-0.0005	0.06589
5	0.87485	0.06589	0.87473	-5.432	0.0001	0.06587
6	0.87485	0.06587	0.87488	-5.434	0.0000	0.06588

Per the table above the yield to maturity is 0.06588

References

[1] Gary Schurman, *The Spot Rate Curve,* February, 2019

[2] Gary Schurman, *Newton-Raphson Method For Solving Nonlinear Equations,* October, 2009

Appendix

A. The solution to the following integral is...

$$\int\limits_s^t\left(a\,u^2+b\,u+c\right)\delta u=\left(\frac{1}{3}\,a\,u^3+\frac{1}{2}\,b\,u^2+c\,u\right)\Bigg[_s^t \tag{16}$$

B. The solution to the integral in Equation (16) above with a lower bound of zero (i.e. $s=0$) is...

$$\left(\frac{1}{3}\,a\,u^3+\frac{1}{2}\,b\,u^2+c\,u\right)\Bigg[_0^t=\frac{1}{3}\,a\,t^3+\frac{1}{2}\,b\,t^2+c\,t \tag{17}$$

C. The solution to the following integral is...

$$\int\limits_s^t\text{Exp}\left\{-\omega\,u\right\}\delta u=-\frac{1}{\omega}\,\text{Exp}\left\{-\omega\,u\right\}\Bigg[_s^t \tag{18}$$

D. The solution to the integral in Equation (18) above with a lower bound of zero (i.e. $s = 0$) is...

$$\int_0^t \mathrm{Exp}\left\{-\omega\, u\right\}\delta u = -\frac{1}{\omega}\left(\mathrm{Exp}\left\{-\omega\, t\right\} - 1\right) = \frac{1}{\omega}\left(1 - \mathrm{Exp}\left\{-\omega\, t\right\}\right) \tag{19}$$

E. The solution to the following derivative is...

$$\frac{\delta}{\delta\omega}\left(\frac{\phi}{\omega}\right) = -\frac{\phi}{\omega^2} \tag{20}$$

F. The solution to the following derivative is...

$$\frac{\delta}{\delta\omega}\left(\mathrm{Exp}\left\{-\omega\, T\right\}\right) = -\omega\,\mathrm{Exp}\left\{-\omega\, T\right\} \tag{21}$$

G. The solution to the following derivative is...

$$\begin{aligned}\frac{\delta}{\delta\omega}\left(\frac{\phi}{\omega}\,\mathrm{Exp}\left\{-\omega\, T\right\}\right) &= -\frac{\phi}{\omega^2}\,\mathrm{Exp}\left\{-\omega\, T\right\} + -\omega\,\mathrm{Exp}\left\{-\omega\, T\right\}\frac{\phi}{\omega} \\ &= -\frac{\phi}{\omega^2}\,\mathrm{Exp}\left\{-\omega\, T\right\} - \phi\,\mathrm{Exp}\left\{-\omega\, T\right\} \\ &= -\phi\,\mathrm{Exp}\left\{-\omega\, T\right\}\left(1 + \frac{1}{\omega^2}\right) \end{aligned} \tag{22}$$